Review: Sequence Convergence - 10/5/16

1 Facts about Converging Sequences

A monotone, bounded sequence always converges.

An unbounded sequence does not converge.

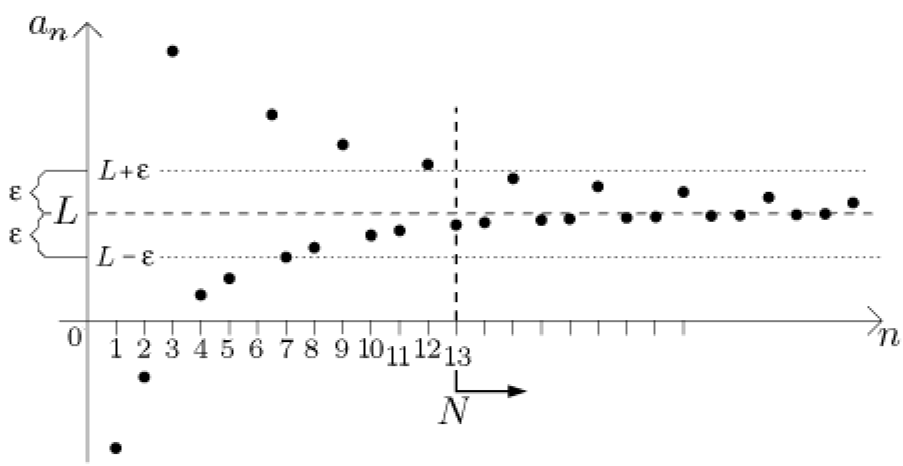
Another way of saying this is: **Convergent sequences are bounded.**

2 Definition of Convergence

Definition 2.0.1 *The **limit** of a sequence $\{a_n\}$ is L if for every $\varepsilon > 0$ there exists a natural number N such that for all $n \geq N$,*

$$|a_n - L| < \varepsilon.$$

What does this mean? First, I'm picking a number $\varepsilon > 0$, and I want to look at a little window around L of size ε. (See picture). The sequence converges if I can find a number N such that all terms after a_N are inside that window.



3 Solving Examples

Example 3.0.2 *Prove that $\lim_{n\to\infty} \frac{1}{n} = 0$. We need to show that there is N such that $|\frac{1}{N} - 0| < \varepsilon$. Since $\frac{1}{N}$ will always be positive, I can take off the absolute values to get $\frac{1}{N} < \varepsilon$, so $\frac{1}{\varepsilon} < N$. This means that as long as N is bigger than $\frac{1}{\varepsilon}$, a_N will be in my ε window around L.*

Example 3.0.3 *Prove that $\{\frac{\sin(n)}{n}\}_{n=1}^{\infty}$ converges to 0. We need to solve for N in $|\frac{\sin(N)}{N} - 0| < \varepsilon$. We can take the absolute value off of the $\frac{1}{N}$ since that will always be positive, but $\sin$ changes between positive and negative, so we need to leave the absolute value on that. So we have $\frac{1}{N}|\sin(N)| < \varepsilon$, so $\frac{1}{\varepsilon}|\sin(N)| < N$. But remember that $\sin$ is bounded, i.e. $|\sin(N)| \leq 1$, so we can take $\frac{1}{\varepsilon}|\sin(N)| \leq \frac{1}{\varepsilon} \cdot 1 < N$. This just means that if we take $N > \frac{1}{\varepsilon}$, it will certainly also be bigger than $\frac{1}{\varepsilon}|\sin(N)|$, which is what we need. Thus $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$.*

Practice Problems

1. Prove that $\lim_{n\to\infty} \frac{3n+1}{n-1} = 3$.
2. Prove that $\lim_{n\to\infty} \frac{3n+1}{n-1} \neq 0$.
3. Prove that $\lim_{n\to\infty} \ln(1+\frac{1}{n}) = 0$.

Solutions

1. $|\frac{3N+1}{N-1} - 3| < \varepsilon$, so combining the fractions give $|\frac{3N+1-3N+3}{N-1}| < \varepsilon$. Provided that $N > 1$, we can take the absolute value off to get $\frac{4}{N-1} < \varepsilon$, so $\frac{4}{\varepsilon} + 1 < N$. Since we have found a formula for N, we have proven that it converges.
2. If we let $\varepsilon = \frac{1}{2}$, then we just have to show that $\frac{3n+1}{n-1}$ is not always within $\frac{1}{2}$ of zero. But $\frac{3n+1}{n-1}$ is actually NEVER within $\frac{1}{2}$ of zero (try plotting the graph to convince yourself of this).
3. $|\ln(1+\frac{1}{N}) - 0| < \varepsilon$. When $x > 1$, then $\ln(x)$ is positive, so we can take off the absolute values to get $\ln(1+\frac{1}{N}) < \varepsilon$. Raising e to both sides gives $1 + \frac{1}{N} < e^{\varepsilon}$, so $\frac{1}{N} < e^{\varepsilon} - 1$. Note that ε is positive, so $e^{\varepsilon} > 1$, so $e^{\varepsilon} - 1$ is positive. Thus we can divide by it without changing the direction of the inequality. Then we have $\frac{1}{e^{\varepsilon}-1} < N$.